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Unstable manifolds of delay systems

Bernd Krauskopf and Kirk Green

Abstract

We present a general method to compute one-dimensional unstable manifolds of saddle points of a Poincaré map of a delay differential equation. This is illustrated with the break-up of a torus in a semiconductor laser with phase-conjugate optical feedback.

In many areas of science one finds models featuring a delay, for example, in biology [11], control theory [3] and laser physics [9]. This leads to a mathematical description by a delay differential equation (DDE) which in its simplest form of a single fixed delay $\tau \in \mathbb{R}$ takes the form:

$$\frac{dx(t)}{dt} = F(x(t), x(t - \tau), \lambda) \quad (1)$$

where

$$F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$$

is differentiable and $\lambda \in \mathbb{R}^p$ is a multi-parameter.

We briefly recall from the theory of DDEs what is needed here; see [1, 6] for details. The *phase space* of (1) is the infinite-dimensional space of continuous functions \mathcal{C} with values in \mathbb{R}^n . The space \mathbb{R}^n is called the *physical space*. A *point* $q \in \mathcal{C}$ is a continuous function

$$q : [-\tau, 0] \rightarrow \mathbb{R}^n .$$

We call $q(0)$ the *head* of q and $\{q(t) \mid t \in [-\tau, 0]\}$ its *history*. The evolution of a point $q \in \mathcal{C}$ after time $t \geq 0$ is given by the *evolution operator*

$$\Phi^t : \mathcal{C} \rightarrow \mathcal{C} .$$

A *solution* of (1) is a function

$$x : [0, \infty) \rightarrow \mathbb{R}^n , \quad t \mapsto \Phi^t(x_0)$$

for some initial point $x_0 \in \mathcal{C}$.

After fixing a suitable section $\Sigma \in \mathbb{R}^n$ (locally) transverse to the flow, the *Poincaré map* P is defined on the space \mathcal{C}_Σ of points in \mathcal{C} with headpoints in Σ as

$$P : \mathcal{C}_\Sigma \rightarrow \mathcal{C}_\Sigma, \quad q \mapsto \Phi^{t_q}(q)$$

where t_q is the return time to Σ . A *periodic orbit* is a solution Γ such that $\Phi^T(q) = q$ for some period $T > 0$ and all $q \in \Gamma$. After choosing a section $\Sigma \in \mathbb{R}^n$ (locally) transverse to Γ , the point $q = \Gamma \cap \mathcal{C}_\Sigma$ is a fixed point of the corresponding Poincaré map P ; see Fig. 1(a). The stability of Γ is given by its *Floquet multipliers*, which are the eigenvalues of the linearization of P at the corresponding fixed point q . It is a crucial property of DDEs that for any fixed radius $r > 0$ there is only a finite number of Floquet multipliers outside a circle of radius r , so that there are always finitely many unstable eigendirections (associated with Floquet multipliers outside the unit circle). As usual, a periodic point is called *hyperbolic* if there are no Floquet multipliers on the unit circle.

We are interested in the one-dimensional unstable manifold $W^u(q)$ of a saddle point q , a hyperbolic fixed point under P with exactly one Floquet multiplier outside the unit circle. The unstable manifold $W^u(q)$ is the set of all points $p \in \mathcal{C}_\Sigma$ that can be iterated backwards under P and are such that $P^l(p) \rightarrow q$ as $l \rightarrow -\infty$. The manifold $W^u(q)$ is tangent at q to the linear unstable eigendirection $E^u(q)$ (which is one-dimensional in our case).

The algorithm

The method we present here in the context of DDEs is based on the algorithm presented in Ref. [10] for computing 1D unstable manifolds in finite-dimensional maps. We grow the manifold as a sequence of points, where the distance between points is governed by the curvature of the *trace* $W^u(q) \cap \Sigma$ of $W^u(q)$ in Σ . The exposition here is necessarily brief, and we refer to Ref. [8] for more details. Throughout we use for illustration the example of a semiconductor laser with phase-conjugate feedback, introduced in the next section.

By using the continuation package DDE-BIFTOOL [2] one can find a periodic orbit Γ with exactly one Floquet multiplier outside the unit circle. From this periodic orbit we find the associated saddle fixed point q after specifying the section Σ ; see Fig. 1(a). The unstable eigendirection $E^u(q)$ can be found with DDE-BIFTOOL or by an iterative power method; see [8] for details. In this way we obtain the necessary starting data, namely the saddle point $q \in \mathcal{C}_\Sigma$ together with its linear 1D unstable eigendirection $E^u(q)$; see Fig. 1(b).

The 1D unstable manifold $W^u(q)$ is represented by a sequence M of points in \mathcal{C}_Σ , with linear interpolation between consecutive list elements. The first point p_1 in this list simply lies at a short distance δ from q . Suppose that we have computed the manifold up to the point p_k . The idea is to find the next point p_{k+1} at some distance Δ_k from p_k ; see Fig. 2. To this end we find the pre-image \hat{p} of p_{k+1} in M , which lies on $W^u(q)$ to good approximation because of the invariance of $W^u(q)$. We first identify the two points p_l and p_{l+1} between which \hat{p} must lie and then \hat{p} itself by bisection in the linear segment between p_l and p_{l+1} . We remark that we allow for a pre-specified tolerance ε such that

$$(1 - \varepsilon)\Delta_k < \|P(\hat{p}(0)) - p_k(0)\| < (1 + \varepsilon)\Delta_k$$

to reduce the number of bisection steps.

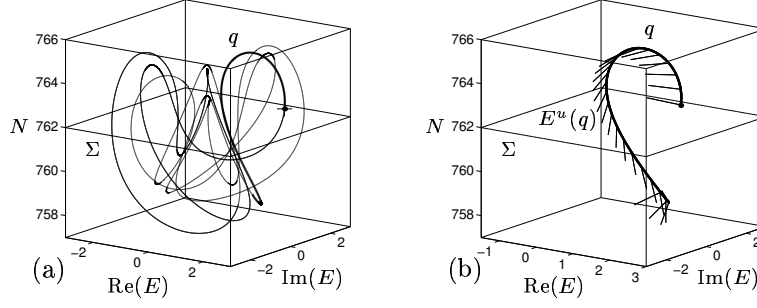


Figure 1: A periodic orbit of the DDE with a periodic point q of the Poincaré map P on it (a), and the linear 1D unstable eigendirection $E^u(q)$ (b).

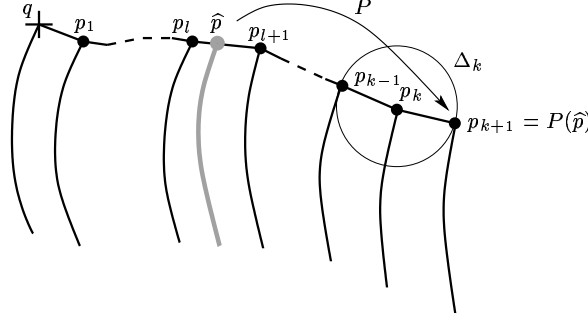
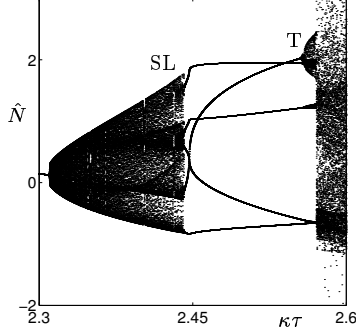


Figure 2: Sketch of the algorithm.

The distance Δ_k is adapted during the computation according to the curvature of the trace $W^u(q) \cap \Sigma$. This involves setting four pre-specified accuracy parameters as is detailed in [8]. It is important to use as few points as possible to achieve a good representation of the trace, because each point in the infinite-dimensional space \mathcal{C}_Σ needs to be represented on the computer by a large array of double precision numbers (each a point in the physical space). In our computations of the PCF laser (introduced below) the array length was 2500, corresponding to an integration time step of $\tau/2500$.

The algorithm stops after a prescribed arclength of the trace has been reached, or when convergence to an attracting fixed point is detected. The latter is inferred when Δ_k drops below a pre-specified small value.

Figure 3: The bifurcation diagram of (2) for $\kappa\tau \in [2.3, 2.6]$.

Break-up of a torus in a PCF laser

We now demonstrate the performance of the algorithm with the example of the break-up of a torus in a semiconductor laser subject to phase-conjugate feedback (PCF). This laser system is described well by the delay differential equations

$$\begin{aligned} \frac{dE(t)}{dt} = & \frac{1}{2} \left[-i\alpha G_N(N(t) - N_{\text{sol}}) + \left(G(t) - \frac{1}{\tau_p} \right) \right] E(t) \\ & + \kappa E^*(t - \tau) \end{aligned} \quad (2)$$

$$\frac{dN(t)}{dt} = \frac{I}{q} - \frac{N(t)}{\tau_e} - G(t) |E(t)|^2$$

for the evolution of the slowly varying complex electric field $E(t) = E_x(t) + iE_y(t) \in \mathbb{C}$ and the population inversion $N(t) \in \mathbb{R}$; see Refs. [4, 7]. Nonlinear gain is included as $G(t) = G_N(N(t) - N_0)(1 - \epsilon P(t))$, where $\epsilon = 3.57 \times 10^8$ and $P(t) = |E(t)|^2$. Note that system (2) is symmetric under the transformation $(E, N) \rightarrow (-E, N)$ so that every invariant set either has this symmetry or has a symmetric counterpart.

After fixing the various laser parameters in (2), including the delay time τ , to realistic values as in Refs. [4, 5, 7], we consider changes of the dynamics upon changing the dimensionless bifurcation parameter $\kappa\tau$. We concentrate on the region shown in Fig. 3 where we integrated (2) and plotted, after transients died away, a normalised value \hat{N} of the inversion whenever the power $P(t)$ crosses its average value. The scenario in Fig. 3 is that of quasiperiodic or locked dynamics on an invariant torus. The dynamics on the torus then locks at the saddle-node of limit cycle bifurcation SL at $\kappa\tau \approx 2.441$ to a periodic orbit that makes five rounds around the torus. This new periodic orbit then bifurcates to a hose-like torus at the torus (or Neimark-Sacker) bifurcation T at $\kappa\tau \approx 2.556$. Finally, there is a sudden transition to chaos at $\kappa\tau \approx 2.571$ in a crisis bifurcation.

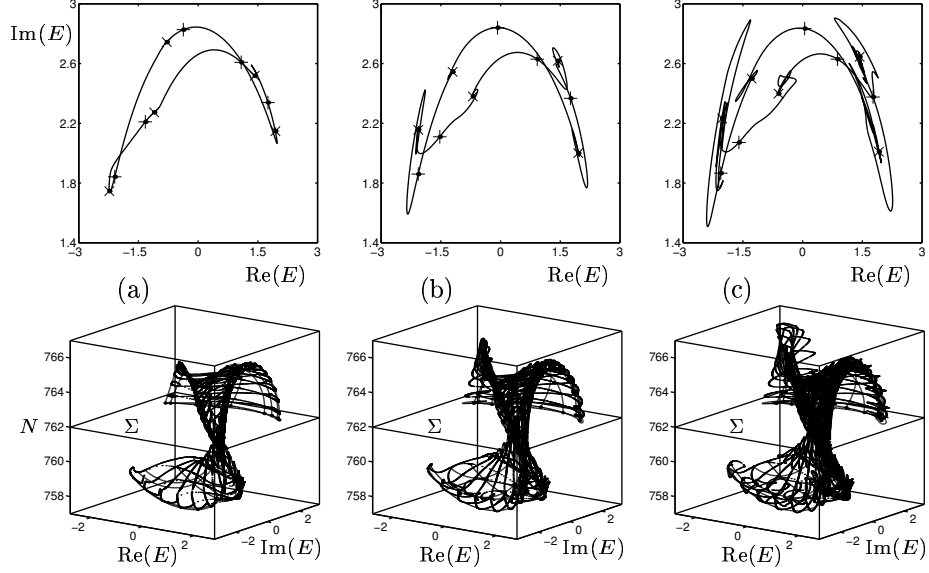


Figure 4: The traces in Σ of the unstable manifolds of five saddle points (top row), and the projection of the manifolds (including the histories of computed points) into (E, N) -space (bottom row); from (a) to (c) $\kappa\tau$ takes the values 2.445, 2.480 and 2.500.

Computing 1D unstable manifolds of the saddle periodic orbit (not shown in Fig. 3) allows us to say much more precisely what is going; see [5] for more details. In Fig. 4 we show the break-up of this torus merely as an illustration of our method. The saddle periodic orbit was computed by continuation with the package DDE-BIFTOOL and the linear unstable eigendirection with an iterative power method.

After the onset of locking, the attracting and the saddle periodic orbit intersect the section $\Sigma = \{(E, N) \mid N = 762\}$ locally in five different points. The torus itself is the closure of the unstable manifolds of the associated five saddle fixed points of the Poincaré map, which is the 5th (local) return map to Σ . As is shown in the top row of Fig. 4, the torus is initially normally hyperbolic, as is expected from general theory. However, for larger $\kappa\tau$ the torus loses its normal hyperbolicity when the manifolds start to spiral into the attracting fixed points. When $\kappa\tau$ is increased further the manifolds make larger and larger ‘excursions’ before ending up at the attracting fixed points. This precedes a crisis bifurcation to chaos; not shown in Fig. 4 and detailed in Ref. [5].

The bottom row of Fig. 4 shows the manifolds including all histories of computed points in projection onto (E, N) -space as a reminder that they indeed lie in an infinite-dimensional phase space. Projection is also the reason that the trace of a 1D unstable manifold may self-intersect and have isolated points where it is not a smooth 1D curve.

In summary, our algorithm can be used in much the same way as methods to compute 1D manifolds in planar diffeomorphisms. This makes it possible for the first time to study global bifurcations in DDEs.

References

- [1] O. Diekmann, S.A. Gils, S.M. Verduyn Lunel, and H.O. Walther, *Delay Equations: Functional-, Complex-, and Nonlinear Analysis*, **110**, *Applied Mathematical Sciences*, (Springer-Verlag, 1995)
- [2] K. Engelborghs, “DDE-BIFTOOL: a Matlab package for bifurcation analysis of delay differential equations” (2000), <http://www.cs.kuleuven.ac.be/~koen/delay/ddebiftool.shtml>
- [3] H. Glusing-Luerssen, “A behavioural approach to delay-differential systems”, *SIAM J. Control Optim.* **35** (1997) 480
- [4] G.R. Gray, D. Huang, and G.P. Agrawal, “Chaotic dynamics of semiconductor lasers with phase-conjugate feedback”, *Phys. Rev. A* **49** (1994) 2096
- [5] K. Green, B. Krauskopf and K. Engelborghs, “Bistability and boundary crisis in a semiconductor laser with phase-conjugate feedback”, to appear in *Physica D*
- [6] J.K. Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, (Springer-Verlag, 1993)
- [7] B. Krauskopf, G.R. Gray, and D. Lenstra, “Semiconductor laser with phase-conjugate feedback: Dynamics and bifurcations”, *Physical Review E* **58**(6) (1998) 7190
- [8] B. Krauskopf and K. Green, “Computing unstable manifolds of periodic orbits in delay differential equations”, to appear in *J. Comp. Phys.*
- [9] B. Krauskopf and D. Lenstra, editors, *Fundamental Issues of Nonlinear Laser Dynamics*, AIP Conference Proceedings **548** (2000)
- [10] B. Krauskopf and H.M. Osinga, “Growing 1D and quasi-2D unstable manifolds of maps”, *J. Comp. Phys.* **146** (1998) 404
- [11] J.D. Murray, *Mathematical Biology*, Biomathematics Texts **19**, (Springer-Verlag, Berlin, 1980)